

Conformal theories with an infrared cutoff

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We give a new perspective on the dynamics of conformal theories realized in the $SU(N)$ gauge theory, when the number of flavors N_f is within the conformal window. Motivated by the renormalization group argument on conformal theories with a finite IR cutoff Λ_{IR} , we conjecture that the propagator of a meson $G_H(t)$ on a lattice behaves at large t as a power-law corrected Yukawa-type decaying form $G_H(t) = \tilde{c}_H \exp(-\tilde{m}_H t)/t^{\alpha_H}$ instead of the exponentially decaying form $c_H \exp(-m_H t)$, in the small quark mass region where $m_H \leq c\Lambda_{\text{IR}}$; m_H is the mass of the ground state hadron in the channel H and c is a constant of order 1. The transition between the “conformal region” and the “confining region” is a first order transition. Our numerical results verify the predictions for the $N_f = 7$ case and the $N_f = 16$ case in the $SU(3)$ gauge theory with the fundamental representation.

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Conformal field theories are ubiquitous in nature and play important roles not only in particle physics beyond the standard model but also in condensed matter physics. Nonperturbative understanding of their dynamics in four-dimensional space-time is ardently desired. In this article, we give a new perspective on the dynamics of conformal theories realized in the $SU(N)$ gauge theory when the number of flavors N_f is within the conformal window [1] by studying meson propagators with a finite infrared (IR) cutoff. Some preliminary results have been presented in Ref. [2].

Our general argument that follows can be applied to any gauge theories with arbitrary representations as long as they are in the conformal window, but to be specific, we focus on $SU(3)$ gauge theories with N_f fundamental fermions (“quarks”). We define the conformal field theory in a constructive way [3]: We employ the Wilson quark action and the standard one-plaquette gauge action on the Euclidean lattice of the size $N_x = N_y = N_z = N$ and $N_t = rN$ with aspect ratio r . We impose periodic boundary conditions except for an anti-periodic boundary condition in the time direction for fermion fields. We eventually take the continuum limit by sending the lattice space $a \rightarrow 0$ with $N \rightarrow \infty$ keeping $L = Na$ fixed. When L is finite, the continuum limit defines a theory with an IR cutoff. The theory is defined by two parameters; the bare coupling constant g_0 and the bare degenerate quark mass m_0 at ultraviolet (UV) cutoff. We also use, instead of g_0 and m_0 , $\beta = 6/g_0^2$ and $K = 1/2(m_0 a + 4)$. For a later purpose, we define the quark mass m_q through Ward-Takahashi identities with renormalization constants being suppressed [2].

Let us quickly remind ourselves of the renormalization group (RG) flow when N_f is in the conformal window. One important fact is finite size lattices in computer simulation always introduce an IR cutoff $\Lambda_{\text{IR}} \sim 1/(Na)$. If the IR cutoff were zero, when quarks have tiny masses, the RG

trajectory would stay close to the critical line, approaching the IR fixed point and finally would pass away from the IR fixed point to infinity. Therefore the IR behavior is governed by the “confining region.” Only on the massless quark line the scale invariance is realized at the IR fixed point. See the left panel of Fig. 1.

When the cutoff Λ_{IR} is finite, the RG flow from UV to IR does stop evolving at the scale Λ_{IR} . When the typical mass scale (e.g., that of a meson) m_H is smaller than Λ_{IR} , it is in the “conformal region.” On the other hand, when m_H is larger than Λ_{IR} , the flow passes away from the IR fixed point to infinity with relevant variables integrated out, thus being in the confining region. See the right panel of Fig. 1.

This scenario implies that when physical quantities at IR (e.g., hadron masses) are mapped into a diagram in terms of physical parameters at UV (e.g., the bare coupling constant and the bare quark mass), there will be gaps in the physical quantities along the boundary between the two phases. There the phase transition will be a first order transition.

To make the above RG argument more concrete, we will study the propagator of the local meson operator

$$G_H(t) = \sum_x \langle \bar{\psi} \gamma_H \psi(x, t) \bar{\psi} \gamma_H \psi(0) \rangle. \quad (1)$$

When the theory is in the confining region, it decays exponentially at large t as

$$G_H(t) = c_H \exp(-m_H t) \quad (2)$$

due to the physical one-particle pole, where m_H is the mass of the ground state hadron in the channel H .

In contrast, we claim that in the conformal region defined by

$$m_H \leq c\Lambda_{\text{IR}}, \quad (3)$$

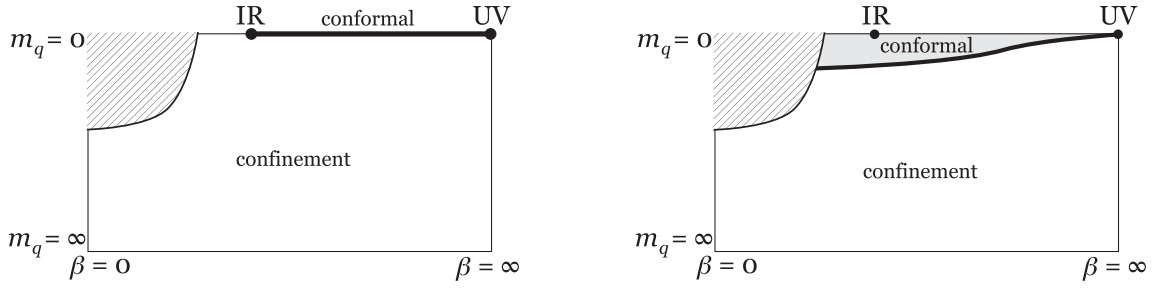


FIG. 1. The phase diagram predicted from the RG argument: (left: for $\Lambda_{\text{IR}} = 0$) and (right: for $\Lambda_{\text{IR}} = \text{finite}$). The shaded strong coupling region for small quark masses does not exist in the $\beta - m_q$ plane [2].

where c is a constant of order 1 which we will determine, the propagator $G(t)$ behaves at large t as

$$G_H(t) = \tilde{c}_H \frac{\exp(-\tilde{m}_H t)}{t^{\alpha_H}}, \quad (4)$$

which is a power-law corrected Yukawa-type decaying form instead of the exponential decaying form [Eq. (2)] observed in the confining region. We further claim that the boundary between the conformal region and the confining region is a first order transition.

We note that the behavior in Eq. (4) is proposed based on the AdS/CFT correspondence with a softwall cutoff in the literature [4]. The meson propagator in the momentum space has a cut instead of a pole: $G_H(p) = 1/(p^2 + \tilde{m}_H^2)^{1-\alpha_H}$. The propagator in the position space (after space integration) takes the form in Eq. (4) in the limit $t\tilde{m}_H \gg 1$.

We distinguish \tilde{m}_H in Eq. (4) from the pole mass m_H in Eq. (2). Equation (3) thus means the lower limit of m_H is $c\Lambda_{\text{IR}}$.

In the continuum limit with $L = \infty$ (i.e., $\Lambda_{\text{IR}} = 0$), the propagator on the massless quark line takes the form

$$G_H(t) = \tilde{c} \frac{1}{t^{\alpha_H}}, \quad (5)$$

consistent with $\tilde{m}_H = 0$ limit of Eq. (4). If we take the coupling constant $g_0 = g^*$ at the UV cutoff, α_H takes a constant value, and the RG equation demands

$$\alpha_H = 3 - 2\gamma^*, \quad (6)$$

for the pseudoscalar (PS) channel with γ^* being the anomalous mass dimension γ at $g = g^*$. The theory is scale invariant (and shown to be conformal invariant within perturbation theory [5]. See also, e.g., Ref. [6], and references therein from AdS/CFT approach). When $0 \leq g_0 < g^*$, α_H depends slowly on t as a solution of the RG equation. In the IR limit $t \rightarrow \infty$, we must retain $\alpha_H(t) \rightarrow 3 - 2\gamma^*$.

The continuum limit with $L = \text{finite}$ defines a continuum theory on $\mathbf{T}^3 \times \mathbf{S}^1$ which corresponds to a compact three-torus at finite temperature. The IR cutoff $\Lambda_{\text{IR}} \sim 1/L$ is finite. The propagator $G_H(t)$ behaves at large t as a power-law corrected Yukawa type decaying form Eq. (4).

The exponent α_H in $t \rightarrow \infty$ with $t\tilde{m}_H \ll 1$ takes the universal formula $3 - 2\gamma^*$ while with $t\tilde{m}_H \gg 1$ it takes a value depending on the dynamics, from which we can estimate the mass anomalous dimensions (see Ref. [2]).

Now let us discuss the results of our numerical calculations. We have performed simulations in the $N_f = 7$ and $N_f = 16$ cases, which we conjecture [7], are the boundaries of the conformal window. The algorithms we employ are the blocked HMC algorithm [8] for $N_f = 2N$ and the RHMC algorithm [9] for $N_f = 1$ in the case $N_f = 2N + 1$.

We specify the coupling constant $\beta = 6.0$ for $N_f = 7$ and $\beta = 11.5$ for $N_f = 16$, taking account of the fact that the IR fixed point for $N_f = 16$ is $\beta = 11.48$ in two-loop approximation. We use the lattices of size $16^3 \times 64$ and $24^3 \times 96$, and vary the hopping parameter K so that the quark mass takes the value from 0.40 to 0.0: We simulate with 15 hopping parameters on the $16^3 \times 64$ lattices, and 5 hopping parameters on the $24^3 \times 96$ lattices.

We choose the run-parameters in such a way that the acceptance of the global metropolis test is about 70%. The statistics are 1,000 MD trajectories for thermalization and 1,000 MD trajectories or 500 MD trajectories for the measurement. We estimate the errors by the jack-knife method with a bin size corresponding to 100 HMC trajectories.

Let us first discuss the results for the $N_f = 7$ case on the $16^3 \times 64$ lattices. We define the effective mass $m_H(t)$ by $\frac{\cosh(m_H(t)(t-N_t/2))}{\cosh(m_H(t)(t+1-N_t/2))} = \frac{G_H(t)}{G_H(t+1)}$. Figure 2 shows the t dependence of the effective mass for the PS channel with three types of sources. On the left panel, we see the clear plateau of the effective mass at $t = 22 \sim 31$ when quark mass is relatively large; $m_q = 0.25$ ($K = 0.1400$). On the other hand, on the right panel, we see the effective mass is slowly decreasing without no plateau up to $t = 31$ when the quark mass is small; $m_q = 0.045$ ($K = 0.1459$), suggesting the power-law correction.

The effective masses for all cases with $m_q \leq 0.172$ ($K \geq 0.1415$) exhibit a similar behavior to the $K = 0.1459$ case. We show the power-law corrected fit for the local-local data in the $K = 0.1459$ case with the fitting range $t = [15:31]$ on the left panel in Fig. 3. The fit with $\alpha_H = 0.8(1)$ reproduces the data very well.

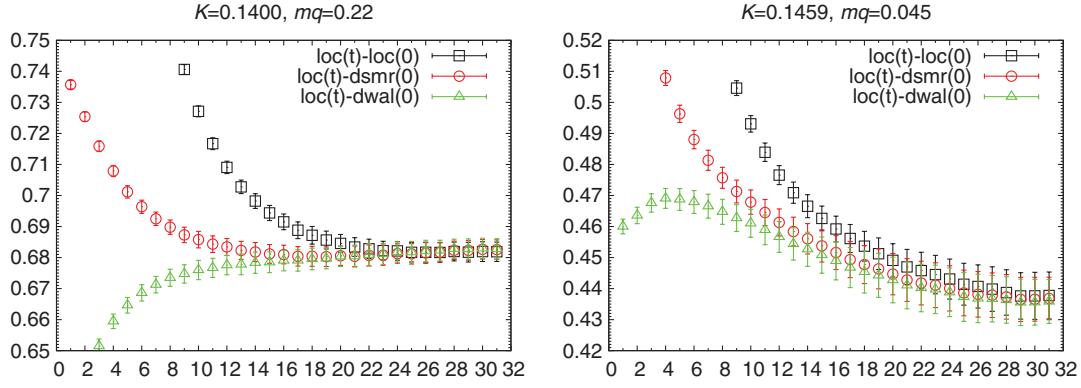


FIG. 2 (color). The effective mass for $N_f = 7$ (left: for $K = 0.1400$; exponential decay) and (right: for $K = 0.1459$; power-law corrected Yukawa decay): With three types of sources; the local-sink local-source (black squares), local-sink doubly-smeared-source (red circles) and local-sink doubly-wall-source (green triangles).

In the case of $N_f = 16$, the effective masses on the $16^3 \times 64$ lattices exhibit the power-law corrected Yukawa-type decaying form for all cases when $m_q \leq 0.0978$ ($K \geq 0.130$). We show the power-law corrected fit for the local-local data in the $K = 0.1315$ case with the fitting range $t = [15:31]$ on the right panel in Fig. 3. The fit with $\alpha_H = 1.15(2)$ reproduces the data well.

We also report that the data on $24^3 \times 96$ lattices for both cases of $N_f = 7$ with $K \geq 0.1459$ and $N_f = 16$ with $K \geq 0.130$ clearly exhibit the power-law corrected Yukawa type decaying form. We obtain $\alpha_H = 0.54(17)$ for $K = 0.1459$ in $N_f = 7$ and $\alpha_H = 1.57(28)$ for $K = 0.130$ in $N_f = 16$, respectively. Thus we have confirmed that the propagators behaves as the power-law corrected Yukawa type decaying form, when the quark mass is small enough. We note in all cases the rho meson mass is degenerate with the pion mass within the one standard deviation. This implies the phase is a chiral symmetric phase.

In the remaining part of the article, we try to identify the boundary between the conformal region and the confining region more precisely by investigating the transition region

carefully with a small step of the value of K for both of the $N_f = 7$ and $N_f = 16$ cases.

We present the results for m_q and m_{PS} (or \tilde{m}_{PS}) in the $N_f = 7$ case in Fig. 4. We first note that the quark mass m_q denoted by black points and line on the left panel is excellently proportional to $1/K$ in the whole region from 0.006 to 0.555.

For the propagators of the PS meson, we observe the clear transition from the exponentially decaying form to the power-law corrected Yukawa-type decaying form at $K = 0.1412 \sim 0.1413$. For $K \leq 0.141$, we present m_{PS} obtained from the fit to the exponentially decaying form with the fitting region $[28:31]$ by green points and line. For $K \geq 0.1415$, we present \tilde{m}_{PS} obtained from the fit to the power-law corrected Yukawa type decaying form with fitting range $[15:31]$ by blue points and line.

The transition region is enlarged on the right panel in Fig. 4. The values of m_{PS} and \tilde{m}_{PS} are different in the limit $K = 0.1412$ from smaller K and from larger K . It should be noted that even if we had ignored the power-law corrections in Yukawa type decay and had estimated the

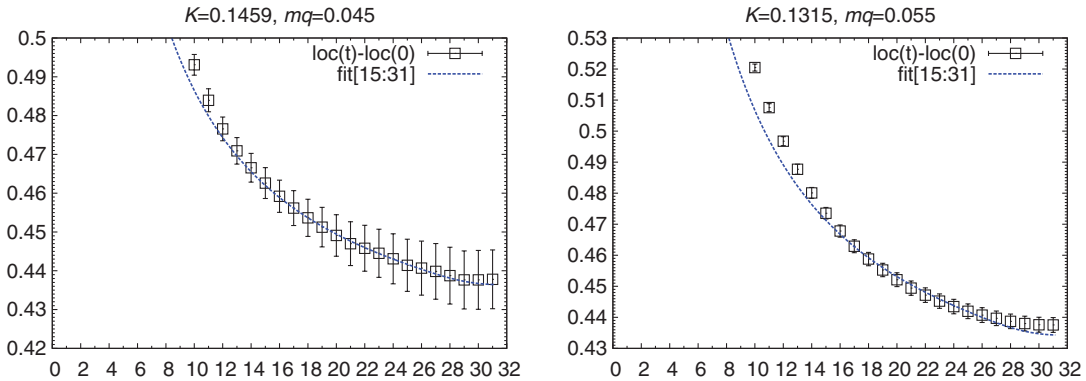


FIG. 3 (color). The effective mass plots for local-sink local-source case and fits by power-law corrected Yukawa type decay (left: for $N_f = 7$ and $K = 0.1459$) and (right: for $N_f = 16$ and $K = 0.1315$).

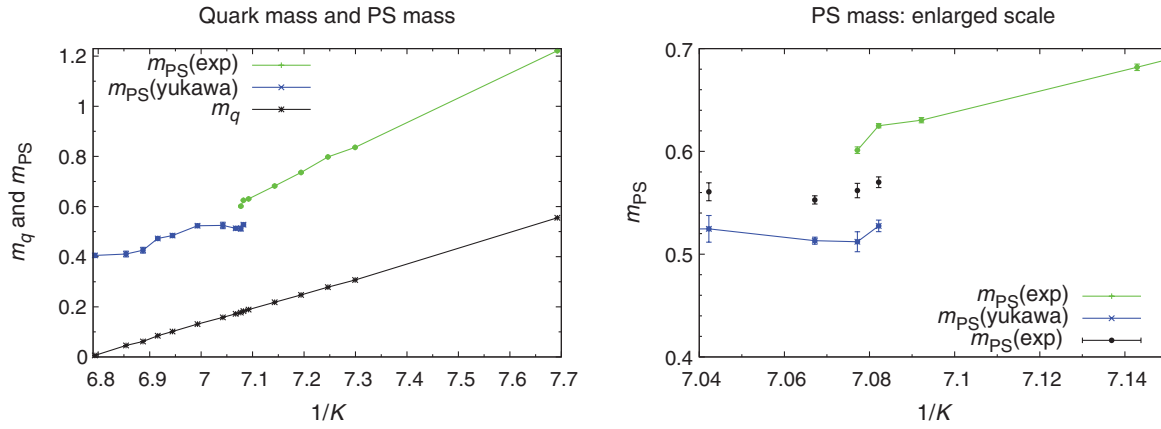


FIG. 4 (color). m_q and m_{PS} (or \tilde{m}_{PS}) vs $1/K$ for the range $0.130 \leq K \leq 0.1472$. The transition region is enlarged on the right panel. See the text for the symbols.

m_{PS} (plotted in Fig. 4 by black points) by assuming exponential decay with the fitting range [28:31], although no plateau is seen as shown on the right panel of Fig. 2 and in Fig. 3, there would have been a gap at the transition point.

It is worthwhile to stress that at $K = 0.1412$ and 0.1413 we observe two states, depending on the initial state. The existence of two states persists at least with 500 MD trajectories. We also note that the exponent α_H does not vanish toward the transition point; it rather increases up to $\alpha_H = 1.1 \sim 1.3$. Thus the transition is a first order transition at $K = 0.1412 \sim 0.1413$. These facts imply that there indeed exists a conformal region, which is separated by a gap of physical quantities from the confining region at finite quark mass.

The m_q dependence of \tilde{m}_{PS} is rather complicated. Once \tilde{m}_{PS} becomes small at the transition point, it slightly increases as m_q decreases, and then decreases at $K \leq 0.143$. Apparently the small \tilde{m}_{PS} region suffers from finite size effects. To verify the scaling relation [10] for \tilde{m}_{PS} in terms of m_q we have to control finite size effects.

Taking $m_{PS} = 0.601 \sim 0.625$ with $m_q = 0.177 \sim 0.183$ ($K = 0.1413 \sim 0.1412$), as the critical mass in Eq. (3), we estimate $c = 2.16 \sim 2.25$, with our working definition of $\Lambda_{IR} = 2\pi(N^3 \times N_t)^{-1/4}$.

In the $N_f = 16$ case we have observed similar results. The m_q dependence of \tilde{m}_{PS} is similarly complicated. The transition occurs at $m_{PS} = 0.513 \sim 0.539$ with $m_q = 0.237 \sim 0.244$ ($K = 0.1255 \sim 0.125$) from which we estimate $c = 1.85 \sim 1.94$.

In the cases of $24^3 \times 96$ lattices for both $N_f = 7$ and $N_f = 16$ cases, we have also observed the transition: In $N_f = 7$, we find exponential decay for $m_q \geq 0.061$

($K \leq 0.1452$) and power-law corrected Yukawa type decay for $m_q \leq 0.045$ ($K \geq 0.1459$). In $N_f = 16$, we find power-law corrected Yukawa type decay for $m_q \leq 0.0978$ ($K \geq 0.130$). The critical mass decreases compared with the $16^3 \times 64$ lattices as expected since the IR cutoff decreases.

We conclude the paper with two main goals achieved in our study. We have given evidence that (1): The meson propagator shows a transition from an exponentially decaying form to a power-law corrected Yukawa-type decaying form at the critical hadron mass whose value is specified by the IR cutoff. (2): The transition is first order. The existence of the conformal region we proposed gives a necessary (but not sufficient) condition for the conformal window, and the result for $N_f = 7$ is consistent with our conjecture that $N_f = 7$ is within the conformal window. However we understand that we need more evidence to conclude that.

In an accompanying publication, we will discuss the physical interpretation of the power-law corrections in terms of the dynamics of the underlying conformal field theory by taking the continuum limit. We also hope to report the physical applications of our findings in the near future.

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